also for a compressible fluid.

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DEVELOPMENT OF THE TOLLMIN - SCHLICHTING WAVE IN A BOUNDARY LAYER, ON A PLATE*

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The development of three-dimensional perturbations of constant frequency in a boundary layer on a semi-infinite plate is studied within the framework of the Navier-Stokes (NS) equations for an incompressible fluid. A case in which the Tollmin-Schlichting (TS) /1, 2/ wave has reached a point on the plate corresponding to the lower branch of the neutral stability curve (NSC), obtained by solving the eigenvalue problem for the Orr-Sommerfeld equation, is discussed. An asymptotic solution of the non-linear NS equations at large Reynolds numbers in given. According to the result obtained, first we have a non-linear process taking place within the NSC near its lower branch, for the separated TS wave with an amptitude that is not too small, leading to gradual reduction in the wave amplitude. Since the Blasius boundary layer is not parallel, the process changes when the amplitude increases. Thus the point at which the amplitude of the TS wave is at a minimum, lies within the loop of the NSC. Therefore, when the experiment is compared with the linear theory based on the Orr-Sommerfeld equation, the theory must be corrected.

Non-linear effects in the theory of the TS waves were first studied in /3/, where an equation for the wave amplitude was given. A strict proof of the amplitude equation was obtained later in /4/ for the case of perturbations periodic in the longitudinal direction of the coordinate. The effect of non-parallelism of the flow on the coefficients of this equation was studied in /5/. The amplitude equation was analysed, without taking into account

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the non-parallel character, in /6/ for the case when the time-dependent frequency of the TS wave did not change, and the change in the amplitude was connected with the motion of the TS wave along the stream. The asymptotic solutions of the complete NS equations were discussed in /7/ for the case when the effects of non-parallelism and non-linearity were both taken into account simultaneously.

The present paper deals with the process of development of the wave when non-linear effects predominate at the initial stage of the process. It is found that the development of the perturbations of the type indicated is described, in the subsequent stage of its evolution, by the asymptotic equations of /7/, and this is proved using the method of matching the asymptotic expansions. In the limit, when the perturbation amplitude is such that the scale of the non-linear and non-parallel development is the same, the results of /7/ are obtained.

1. Formulation of the problem.

Let us consider two-dimensional flow past a finite plate parallel to the uniform incoming flow. We introduce a Cartesian coordinate system with origin at the leading edge and abscissa axis directed along the plate. The subsequent investigation will be carried out at a distance of the order of L_* and such that $\varepsilon = R^{-1/8}$ is a small parameter where $R = U_* L_*/\nu_*$. An asterisk denotes dimensional quantities: ν_* is the coefficient of kinematic viscosity and U_* is the velocity of the incoming flow. We assume that a TS wave is formed on approaching the point L_*, whose amplitude within the main body of the boundary layer formed is of the order of $\varepsilon\delta$ (for the longitudinal velocity u_*), where $\delta \ll 1$, and at the point L_* the increment of the wave growth in the coordinate x_* will vanish in accordance with the linear theory. The structure of the TS wave in the linear formulation was studied in /8-10/. Below we shall consider the change in the oscillation amplitude of the wave within the loop of the NSC near its lower branch, taking into account the effects of the non-linearity and non-parallelism.

According to the linear theory /9/ the amplitude of the TS wave will not change at distances of the order of the corresponding wave length, i.e. at distances $x_1 \equiv (x_* - L_*) L_*^{-1}\varepsilon^{-3} = O(1)$. However, it is necessary, because of the non-linearity, to supplement the term of order $\varepsilon\delta$ (the TS wave) in the expansion for u_* with terms of the order $\varepsilon\delta^2$ with second and zeroth harmonics and of the order $\varepsilon\delta^3$ with third and first harmonics relative to the first harmonic of the initial wave. The higher-order terms are obtained from the projection of the NS equation on the x_* axis. The latter terms, appearing in the same harmonic as the initial wave, will begin to influence the change in the amplitude, and this can be described by assuming that the wave amplitude begins to depend on the variable $X = x_1\delta^2$. At the same time, effects connected with the change in amplitude of the TS wave (in the projection of the NS equations on the x_* axis) will be of the same order as the non-linear effects shown above.

The influence of the non-parallelism already becomes essential for the development of a wave of small amplitude, in the neighbourhood of the point L_*, i.e. when $x \equiv (x_* - L_*)/L_* \ll 1$. According to the selfsimilar Blasius solution describing a flow in a developed boundary layer, the longitudinal velocity profile changes during the displacement by xL_*, by the order of x. In this connection, terms of the order of $\delta x/\varepsilon$ (with the same harmonic as the initial wave) appear in the projection of the NS equations on the x_* axis, together with terms of the order of δ^3/ε, obtained because of the non-linearity, and controlling the change in the amplitude of the initial wave, and this is caused by the deformation of the Blasius profile. Then the condition that at the initial stages of the development of the wave it is the effects of non-linearity and non-parallelism that are important, means that $x \ll \delta^2$. Expressing x in terms of the variable $X = O(1)$ (on the scale of the non-linearity), we obtain the constraints $1 \gg \delta \gg \varepsilon^{3/4}$ which will be assumed hold in what follows.

Let the amplitude of the TS wave at the point x be a quantity of the order of $\varepsilon\delta_0$, where δ_0 depends on x $(\delta_0(X = O(1)) = \delta)$. A "weak" non-linearity of the flow near the point x will manifest itself, as was explained above, in the fact that the perturbation amplitude will be a function, of the order of unity, of the variable $Z = x\delta_0^2/\varepsilon^3$. As the wave moves, the effects of the non-parallelism become stronger and attain the same order as those of the non-linearity when the wave reaches x such as $x = O(\delta_0^2)$ on the scale $Z = O(1)$. Then we can easily obtain $\delta_0 = O(\varepsilon^{3/4})$, $x = O(\varepsilon^{3/2})$ and the corresponding "slow" variable $Z = x/\varepsilon^{3/2}$. We show in this paper that two layers can be introduced when $x > 0$: the first layer will have a size of the order of ε^3/δ^2, i.e. $X = O(1)$, where the non-linear processes are important, and the second layer will be of the order of $\varepsilon^{3/2}$, i.e. $Z = O(1)$ where the effects of the non-parallelism and non-linearity are of the same order and asymptotic matching exists between them.

The problem for the NS equations is formulated as follows. To construct an asymptotic solution in the layer $X = O(1)$, $X > 0$ and a solution in the layer $Z = O(1)$, satisfying the conditions of adhesion at $y_* = 0$ and of decay of the perturbations when $y_*/L_* \to \infty$, such that they match asymptotically as $X \to \infty, Z \to 0$ and yield a TS wave as $X \to 0$, with amplitude of the order of $\varepsilon\delta$ in the basic thickness of the boundary layer.

2. Description of the flow in the viscous boundary layer. Let us separate the plane of flow, as usual /8/, into three regions: the lower region where the viscosity is essential for the perturbations constructed, the middle region which coincides with the main body of the boundary layer, and the upper region of potential flow of which the increase in the ordinate is accompanied by a final exponential decay of the perturbations.

Applying the technique of multiscale expansions, we introduce into each region its own variable in y_*, and the same variables in t_* and x_*, namely the "rapid" variables $t_1 \equiv t_* U_* L_*^{-1} \varepsilon^{-2}$, x_1 and the "slow" variables X, Z.

In order to describe the flow in the boundary region, we introduce a variable $y = y_* L_*^{-1} \varepsilon^{-5}$ which will be of the order of unity. We seek the expansions in this region at $X = O(1)$ for u_*, for the pressure p_* and for transverse velocity v_* in the form

$$\frac{v_*}{U_* \varepsilon^3 \delta} = v_3 + \Delta, \quad \frac{u_* - \lambda_1 U_* \varepsilon y}{U_* \varepsilon \delta} = u_3 + \Delta, \quad \frac{p_* - p_*^\circ}{U_*^2 \rho_* \varepsilon^2 \delta} = p_3 + \Delta \tag{2.1}$$

The quantity $\lambda_1 = 0.3321$ in (2.1) is the derivative of the longitudinal velocity of the Blasius solution for the selfsimilar variable calculated at the plate, p_*^0 is the unperturbed pressure, ρ_* is the density of the incoming flow and Δ is a quantity of the order of $O(\varepsilon) + O(\delta^3)$. We seek the functions u_3, v_3 in the layer $X = O(1)$ in the form

$$\begin{aligned}
&-2v_3/\omega_0 = i f_1 C_1 E + 2i\delta f_2 C_2 E^2 + \delta^2 (3i f_3 C_3 E^3 + i\Phi_1 E) + \text{c.c.} \\
&2u_3 = f_1' C_1 E + \delta (f_0 C_0 + f_2' C_2 E^2) + \delta^2 (f_3' C_3 E^3 + \Phi_1' E - f_1' dC_1/ \\
&/dX \cdot E/\mathrm{i}\, \omega_0) + \text{c.c.} \\
&f_n = f_n(y), \quad \Phi_1 = \Phi_1(x, y), \quad C_n \equiv C_n(X) = (C_1(X))^n, \quad C_0 = \\
&|C_1|^2 \\
&E = \exp(i\omega_0 x_1 + i\omega_1 t_1)
\end{aligned} \tag{2.2}$$

Here C_1 is an unknown complex function (the amplitude of the TS wave), ω_0 and ω_1 are unknown real numbers, the symbol c.c. denotes the complex conjugate and a prime denotes differentiation with respect to y. The equation of continuity holds identically for expressions (2.2).

Substituting the expansions (2.1), (2.2) into the NS equations and collecting the quantities accompanying terms of the same order of smallness and harmonics, we obtain the following systems of equations for determining the functions f_n, C_n, Φ_1, with the boundary conditions of adhesion at $y = 0$ and boundedness as $y \to \infty$:

$$\begin{aligned}
&L_1(f_1) = 0, \quad \Gamma_0(f_1) = 0; \quad L_2(f_2) = {}^1/_2 i\omega_0 \langle f_1'' f_1' \rangle \\
&\Gamma_0(f_2) = 0; \quad f_0''' = -{}^1/_2 \omega_0 \operatorname{Im} \langle f_1'' f_1' \rangle, \; f_0(0) = 0
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&L_1(\Phi_1) = \lambda_1 y f''_1 dC_1/dX + C_1 |C_1|^2 i\omega_0 \langle f_2'' \bar{f}_1' + f_1'' f_0 + \\
&\quad {}^1/_2 f_2''' \bar{f}_1 \rangle, \; \Gamma_0(\Phi_1) = 0
\end{aligned} \tag{2.4}$$

Here

$$\begin{aligned}
&L_n(\Phi) \equiv \Phi'''' - in(\omega_0 \lambda_1 y + \omega_1) \Phi'' \\
&\Gamma_0(\Phi) \equiv |\Phi(0)| + |\Phi'(0)| + |\Phi''(\infty)|, \quad \langle \Phi^{(n)} \Psi^{(m)} \rangle = \\
&\quad \Phi^{(n)} \Psi^{(m)} - \Phi^{(n-2)} \Psi^{(m+2)}
\end{aligned}$$

The numbers n and m in the last formula denote the order of the derivatives, and a bar denotes the complex conjugate.

We seek the solution in the layer $Z = O(1)$ also in the form (2.1), (2.2), but we must replace λ_1 in (2.1) by $\lambda_1 (1 - {}^1/_2 x)$, and δ and X in all formulas by $\varepsilon^{2/4}$ and z respectively. Systems (2.3) are rewritten without any changes, and for system (2.4) we have

$$\begin{aligned}
&L_1(\Phi) = \lambda_1 y f_1'' dC_1/dZ + i\,\omega_0 C_1 |C_1|^2 \langle f_2'' \bar{f}_1' + f_1'' f_0 + \\
&\quad {}^1/_2 f_2''' \bar{f}_1 \rangle - {}^1/_2 Z \lambda_1 i\omega_0 f_1'' y C_1, \; \Gamma_0(\Phi_1) = 0
\end{aligned} \tag{2.5}$$

In the layer $Z = O(1)$ the terms of the order $O(\varepsilon)$ appearing in Δ, and correcting the solution of the Orr-Sommerfeld equation, does not affect the calculation of the amplitude $C_1(Z)$ of the TS wave /7/, and is given e.g. in /11/.

3. Description of the potential flow. In order to describe a flow in the potential region in the layer $X = O(1)$, we introduce the variable $y_1 = y_* \varepsilon^{-3} L_*^{-1}$, which is of the order of unity. We seek the expansion within this region in the form

$$\frac{u_* - U_*}{U_* \varepsilon^2 \delta} = u_1 + \Delta, \quad \frac{v_*}{U_* \varepsilon^2 \delta} = v_1 + \Delta, \quad \frac{p_* - p_*^\circ}{U_*^2 \rho_* \varepsilon^2 \delta} = p_1 + \Delta \tag{3.1}$$

We seek the functions u_1, v_1 in the form

$$2u_1 = g_1'H_1E + \delta(g_2'H_2E^2 + g_0H_0) + \delta^2\Big(g_3'H_3E^3 + G_1'E - g_1'\frac{dH_1}{dX}\frac{E}{i\omega_0}\Big) + \text{c.c.}, \quad \frac{-2v_1}{\omega_0} = ig_1H_1E + \delta 2ig_2H_2E^2 + \delta^2(iG_1E + 3ig_3H_3E^3) + \text{c.c.} \tag{3.2}$$

$$g_n = g_n(y_1), \quad H_n = H_n(X), \quad G_1 = G_1(y_1, X)$$

(a prime denotes differentiation with respect to y_1). From the NS equations it follows that the quantities u_1, p_1, v_1 can be found from the equations

$$\frac{\partial v_1}{\partial x_0} - \frac{\partial u_1}{\partial y_1} = 0, \quad \frac{\partial u_1}{\partial x_0} + \frac{\partial p_1}{\partial x_0} = 0, \quad \frac{\partial v_1}{\partial y_1} + \frac{\partial u_1}{\partial x_0} = 0 \tag{3.3}$$

$$\Big(\frac{\partial}{\partial x_0} \equiv \frac{\partial}{\partial x_1} + \delta^2\frac{\partial}{\partial X}\Big)$$

After substituting the expansions (3.2) into it, system (3.3) yields

$$N_n(g_n) = 0, \quad n = 1, 2, 3, \quad N_1(G_1) = (g_1''/(i\omega_0)) - i\omega_0 g_1)dH_1/dX \tag{3.4}$$

$$(N_n(G) \equiv G'' - n^2\omega_0^2 G)$$

Since $\omega_0 < 0$ (this will be shown below) we must choose, in accordance with the conditions of decay as $y_1 \to \infty$, $g_1 = \exp(\omega_0 y_1)$. Then from the last equation of (3.4) we have the following solution:

$$G_1 = \exp(\omega_0 y_1)(H_{11} - iy_1 dH_1/dX)$$

where $H_{11} = H_{11}(X)$ is an arbitrary complex function determined from the conditions of matching the solutions in three regions. Let us write v_1, u_1 in the form

$$2v_1 = \sum_{n=0}^{3} v_{1n}E^n + \text{c.c.} \qquad 2u_1 = \sum_{n=0}^{3} u_{1n}E^n + \text{c.c.}$$

Substitution of the solution (3.4) into the expansions (3.2) makes it possible to determine v_{1n}, u_{1n} and establish a relation between them, which also holds for the layer $Z = O(1)$

$$v_{1n}(y_1 = 0) + iu_{1n}(y_1 = 0) = 0 \quad (n \text{ is a natural number}) \tag{3.5}$$

We seek the solution in the layer $Z = O(1)$ in the given region in the form (3.1), (3.2), with δ replaced by $\varepsilon^{3/4}$, X and Z.

4. **Basic part of the boundary layer.** As was showed earlier, in the basic part of the boundary layer the variable $y_2 \equiv y_1/\varepsilon = O(1)$. We seek the expansion in this region in the form

$$\frac{u_* - U_*U(y_2)}{U_*\varepsilon\delta} = u_2 + \Delta, \quad \frac{v_*}{U_*\varepsilon^2\delta} = v_2 + \Delta, \quad \frac{p_* - p_*^\circ}{U_*^2\rho_*\varepsilon\delta} = p_2 + \Delta \tag{4.1}$$

where $U(y_2)$ is an arbitrary Blasius velocity profile at $x = 0$. The functions u_2, p_2, v_2 depend on x_1, t_1, X, y_2. Substituting (4.1) into the NS equations, we obtain the following system for the principal terms in ε:

$$U(y_2)\frac{\partial u_2}{\partial x_0} + v_2\frac{dU}{dy_2}(y_2) = 0, \quad \frac{\partial p_2}{\partial y_2} = 0, \quad \frac{\partial u_2}{\partial x_0} + \frac{\partial v_2}{\partial y_2} = 0$$

which can be solved explicitly

$$u_2 = A\frac{dU}{dy_2}(y_2), \quad p_2 = p_2(x_1, t_1, X), \quad A = A(x_1, t_1, X), \quad v_2 = -\frac{\partial A}{\partial x_0}U(y_2) \tag{4.2}$$

The arbitrary function p_2 and A can be found as a result of matching.

The conditions of matching the transverse velocity and pressure during the passage from the region of boundary layer to the potential flow, are

$$v_1(y_1 = 0) = -\partial A/\partial x_0, \quad p_1(y_1 = 0) = p_2 \tag{4.3}$$

The conditions of matching the longitudinal velocity and pressure during the passage from the region of viscous sublayer to the main bulk of the boundary layer, are

$$u_3(y = \infty) = \lambda_1 A, \quad p_3 = p_2 \tag{4.4}$$

Now, using the relations (4.3), (4.4), (3.3), (3.5), (2.2) and condition

$$\partial p_3/\partial x_0 = \partial^2 u_3/\partial y^2 \ (y = 0)$$

we can obtain

$$\Gamma_1(f_1) = 0,\ \Gamma_1(\Phi_1) = -2\omega_0 f_1'(\infty)\, dC_1/dX,\ f_0'(\infty) = 0 \qquad (4.5)$$
$$\Gamma_2(f_2) = 0,\ \Gamma_n(\Phi) \equiv \lambda_1\Phi'''(0) + i\omega_0^2 n^2 \Phi'(\infty)$$

We seek the solution in the layer $Z = O(1)$ in the form (4.1) where δ has been replaced by $\varepsilon^{3/4}$, X, by Z, $U(y_2)$ by $U(y_2)(1 - {}^1/_2 x y_2 d\ln U/dy_2)$ (the last substitution represents the expansion of the Blasius profile in x in the neighbourhood of L_*. Using a procedure analogous to the derivation of (4.5), we can obtain the boundary condition for Φ_1 in the layer $Z = O(1)$

$$\Gamma_1(\Phi_1) = -2\omega_0 f_1'(\infty)\, dC_1/dZ - {}^1/_2 i\omega_0^2 f_1'(\infty) Z C_1 \qquad (4.6)$$

The boundary conditions and equations for f_1, f_0, f_2 remain unchanged.

5. Equation for the amplitude.

Let us now change, in systems (2.3), (2.7) supplemented by the boundary conditions (4.5), (4.6), to new variables

$$f_2 = \lambda_1^{1/2} f_2',\ f_0 = \lambda_1^{5/4} f_0',\ C_1 = \lambda_1^{-1/2} C_1',\ \omega_0 = \lambda_1^{5/4}\omega_{00}$$
$$\omega_1 = \lambda_1^{3/2}\omega_{10},\ y = \lambda_1^{-3/4} y',\ X = \lambda_1^{-3/4} X',\ Z = \lambda_1^{-3/4} Z'$$

The new systems can be obtained from the old ones by replacing ω_0 by ω_{00}, ω_1 by ω_{10}, λ_1 by unity, and the factor Z by $\lambda_1^{-3/4} Z'$, and we shall retain the old numbering. In what follows, we shall omit the primes from the new variables.

We can write systems (2.3), (2.4), and (2.5) in the form $L_{0n}: \overline{\Phi} \to (\psi, \varphi)$, where the operator L_{0n} is given by the relation

$$L_n(\Phi) = \psi,\ \Gamma_n(\Phi) = \varphi,\ \Gamma_0(\Phi) = 0$$

The kernel of the operator L_{0n} is identically zero when $n \neq 1$, i.e. an inverse operator L_{0n}^{-1} exists and the kernel represents, for $n = 1$, the integrals of the Airy function $\mathrm{Ai}(\Omega)$. A solution will exist when $n = 1$ if and only if the following relation holds ($\mathrm{Bi}(\Omega)$ is a standard special function):

$$P(\psi) \equiv \frac{i\omega_{00}^2\pi}{k_{11}} \int_0^\infty \Big\{\mathrm{Bi}(\Omega_1)\int_\infty^{\xi_1} \mathrm{Ai}(\Omega_2)\Psi(\xi_2)\,d\xi_2 - \mathrm{Ai}(\Omega_1)\int_0^{\xi_1}\mathrm{Bi}(\Omega_2)\times \qquad (5.1)$$
$$\Psi(\xi_2)\,d\xi_2\Big\}\,d\xi_1 - \pi\frac{d\,\mathrm{Bi}}{d\Omega}(k_{12})\int_0^\infty \mathrm{Ai}(\Omega_1)\Psi(\xi_1)\,d\xi_1 = \varphi$$
$$k_{11} = |\omega_{00}|^{1/3}\exp(-i\pi/6),\quad k_{12} = k_{11}\omega_{10}/\omega_{00},\quad \Omega_n = k_{11}\xi_n + k_{12}$$

System (2.3) has a solution under the condition that $\omega_{00} = -1.0005$, $\omega_{10} = 2.298$ /6/. Here $f_1'(y) = K\,\mathrm{Ai}(k_{11}y + k_{12})$ and we choose the constant K, to be specific, equal to $\sqrt{\pi}$ (this will affect only the initial conditions for C_1 when $X = 0$). Then the expansions (2.1), (2.2) will represent, to the first order, the neutral TS wave.

The condition for (2.4) to be solvable (with the corresponding condition from (4.5) for the layer $X = O(1)$) yields

$$(2\omega_{00}f_1'(\infty) + P(yf_1''))\,dC_1/dX + i\omega_{00}P(\langle f_2''\bar{f}_1' + f_1''f_0 + \tfrac{1}{2}f_2'''\bar{f}_1\rangle)\,C_1|C_1|^2 = 0$$

or

$$dC_1/dX = K_1 C_1|C_1|^2 \qquad (5.2)$$
$$a \equiv \mathrm{Re}\,K_1 = -0.4099,\ r_1 \equiv \mathrm{Im}\,K_1 = 0.4533$$

Let $f = |C_1|$. Then $f(X) = (f_{(0)}^{-2} - 2aX)^{-1/2}$, where $f(0) = O(1)$ is the parameter of the problem. Since $a < 0$, we find that the amplitude of the TS wave decays when $X > 0$ as $(-2aX)^{-1/2}$ on the scale $X = O(1)$. Then from (5.2) it follows that

$$C_1 \to (-2aX)^{-1/2}\exp(ir_1/(-2a)\ln X + i\varphi_0) \quad \text{when} \quad X\to\infty \qquad (5.3)$$

where φ_0 is an arbitrary real constant determined by the initial conditions when $X = 0$.

The condition for system (2.5) to be solvable for the layer $Z = O(1)$, taking (4.6) into account, yields

$$dC_1/dZ = K_1C_1|C_1|^2 + DZC_1 \qquad (5.4)$$
$$d_1 \equiv \mathrm{Re}\,D = 0.7441,\ s = \mathrm{Im}\,D = 0.2309$$

An analogous equation was obtained in /7/. We find that (5.4) can also be solved explicitly. Let us write $f = |C_1|$, $\Psi = \arg C_1$. Then (5.4) will be equivalent to the system

$$df/dZ = af^3 + d_1Zf,\ d\Psi/dZ = r_1f^2 + sZ \tag{5.5}$$

The substitution $f^2 = \Phi^{-1}$ reduces the first equation of this system to a linear equation whose solution is

$$\Phi = \left\{(-2a)\int_0^Z \exp(d_1\xi^2)\,d\xi + K_0\right\}\exp(-d_1Z^2)$$

where K_0 is an arbitrary positive constant. The function Ψ can then be defined as follows:

$$\Psi = \Psi_0 + \frac{s}{2}Z^2 + r_1\int_0^Z f^2(\xi)\,d\xi \tag{5.6}$$

where Ψ_0 is a real constant. Analysis of the asymptotic expressions of the solution of (5.5) yields

$$f(Z\to 0)\to (K_0 - 2aZ)^{-1/2} + o(1) \tag{5.7}$$
$$f(Z\to\infty)\to\left(-\frac{d_1}{a}\right)^{1/2} Z^{1/2} + o(Z^{1/2})$$

If we continue the solution (2.2), constructed in the layer $X = O(1)$ in terms of its asymptotic forms as $X\to\infty$, to scales of the order of $Z = O(1)$ (this formally reduces to substituting $X = Z\delta^2/\varepsilon^{3/2}$ into (5.3), and then into (2.2)). Then, when $K_0 = 0$ and for the corresponding value of Ψ_0 we obtain the asymptotic expression at the zero $(Z\to 0)$ for the solution in the layer $Z = O(1)$. In the limit, when $\delta^2 \simeq \varepsilon^{3/2}$, the scales of X and Z will be the same and the non-linearly non-parallel development of the perturbations will occur from the very beginning, and this was studied in /7/.

6. Basic results.

Analysis of the asymptotic expressions (5.7) shows that a point Z_0 exists at which the amplitude f in the layer $Z = O(1)$ reaches its minimum m. Numerical calculations on a computer gave $Z_0 = 1.072$, $m = 1.395$. According to the linear theory the following condition holds /10, 11/ for the lower branch of NSC as $\varepsilon\to 0$:

$$F \equiv \omega_*\nu_*/U_*^2 \to \lambda_1^{3/2}\omega_{10}\varepsilon^6 + \Delta_1\varepsilon^7 + O(\varepsilon^8) \tag{6.1}$$

where Δ_1 is a number.

The distance along the plate from its beginning to the point of which a fixed frequency ω_* perturbations has a smallest amplitude at the bottom of the boundary layer, is equal to $L_*^\circ = L_*(1 + Z_0\varepsilon^{3/2})$. The number $\varepsilon_0 = (U_*L_*^\circ/\nu_*)^{-1/8}$ found from this distance is connected with ε by the relation $\varepsilon = \varepsilon_0(1 + Z_0\varepsilon_0^{3/2}/8) + O(\varepsilon_0^{5/2})$. Substituting this relation into (6.1) we obtain, as $\varepsilon_0\to 0$

$$F\to\lambda_1^{3/2}\omega_{10}\varepsilon_0^6(1 + 3Z_0\varepsilon_0^{3/2}/4) + \Delta_1\varepsilon_0^7 + O(\varepsilon_0^8) \tag{6.2}$$

Since the Reynolds number in the experiment is determined from L_*° /2/, comparison with the theory must be made using the formula (6.2).

The proposed theory holds when $1 \gg \delta \gg \varepsilon^{3/4}$, or, using the dimensional coordinates, when

$$1 \gg A_*/\varepsilon U_* \gg \varepsilon^{3/4} \tag{6.3}$$

where A_* is the perturbation amplitude (separated TS wave) for the longitudinal velocity at the bottom of the boundary layer at the point L_*. The wave satisfying condition (6.3) has a minimum amplitude within the NSC loop.

Condition (6.3) facilitates the study of the problems of "susceptibility" /2/ for various types of perturbations leading to the satisfaction of these conditions, since the subsequent stage of development of these perturbations in the layer $Z = O(1)$ in the same with an accuracy of up to the phase Ψ_0 from (5.6), determined by the value $C_1(X = 0)$.

The proposed theory is based on the assumption that during the approach to the point L_* the perturbation takes the form of the TS wave with a zero growth increment. The wave can be generated by, e.g., a vibrator at the point L_* oscillating with critical frequency, with dimensions given in /9/.

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ON FORCED OSCILLATIONS IN THE BOUNDARY LAYER AT FREQUENCIES NEAR THE UPPER BRANCH OF THE NEUTRAL CURVE*

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Perturbations introduced into the boundary layer (BL) of an incompressible liquid by a harmonic oscillator in the form of a moving section of the surface are examined. Outside the oscillating part, the streamlined solid is a plane plate. It is assumed that the Reynolds number is large and the oscillation frequency, corresponds, in order of magnitude, to the asymptotic form of the upper branch of the neutral stability curve (NSC). The system of equations for perturbations, at small amplitudes of the oscillator, is linearized and is solved by the Fourier method. In addition, for each Fourier component, the flow field is divided into five sublayers. The amplitude of a Tollmin-Schlichting wave (TS) is calculated and separated from the perturbed background downstream of the oscillator. If the oscillator frequency exceeds the neutral value at the upper branch of NSC with the given Reynolds number, the TS wave amplitude decays. For frequencies below neutral, the wave amplitude increases exponentially downstream. In the final example, the parameters of the TS wave fall within the unstable region, between two NSC branches.

At a distance L^* from the front edge of the plane surface with an incompressible viscous liquid flowing over it, let there be a moving section of surface of length l^* which is oscillating at a frequency ω^* (henceforth the asterisk denotes dimensional quantities). Defining the Reynolds number as $R = U_\infty^* L^*/\nu^*$, where U_∞^* is the velocity of the oncoming flow and ν^* is the kinematic viscosity, we will assume $R \to \infty$. An investigation of the perturbation propagation process caused by the moving section is one of the problems of BL reproducibility.

The solution of this problem taking compressibility into account and for any Mach number at infinity was obtained previously in /1, 2/ for $l^* = O(R^{-3/8}L^*)$, $\omega^* = O(R^{1/4}U_\infty^* L^{*-1})$, starting from the linearized equations of the theory of free interaction /3, 4/, which, as is well-known /5, 6/, is subject to perturbations in the given range of frequencies. In the subsonic case, the first mode from the spectrum of eigensolutions of these equations corresponds to the

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